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A Comparison between the Method of Integral Relations and the Method of Lines as Applied to the Blunt Body Problem

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THIS Note provides a direct comparison between the well-known method of integral relations[1,2] and the method of lines as applied to the problem of a blunt body with detached shock; the results tend to refute the notion that the method of integral relations is necessarily more accurate when only a few strips are employed.[1] The results of both methods agree favorably with each other and "exact" solutions over a wide range of Mach numbers even for the one-strip approximation used here. In some cases the method of lines even surpasses the method of integral relations in predicting certain aspects of the flow.

Both approaches retain the concept of dividing the subsonic shock layer into strips parallel to the body. The difference arises in transforming the original system of partial differential equations to one involving only ordinary differential equations. As opposed to the method of integral relations, which integrates the equations in the n direction (normal to the body) by assuming some appropriate polynomial expansion, the method of lines approximates the $(\partial/\partial n)(\;)$ terms directly through use of finite differences. In either case the variable values required at each interface become the dependent variables in a system of simultaneous ordinary differential equations in s, the arc length along the body. Physically speaking, the integral relations require that the conservation equations be satisfied in an average manner over the strips; with the method of lines they are satisfied exactly at each interface. Although only one strip was used in this study, the basic limitation on accuracy is the number of strips one is willing to program, since algebraic and computational difficulties rapidly multiply. Here the method of lines has a distinct advantage in that recursive relations can be derived, that allow the computer to perform much of the algebra as shown in Refs. 3 and 4.

In the following, the development as well as the notation closely follows that of Ref. 5, which treats the method of integral relations alone. The equations of motion for a compressible inviscid gas, the perfect gas law, and streamwise entropy conservation behind the shock are employed. Referring to Fig. 1, in body-oriented curvilinear coordinates (s,n) normalized to the body radius of curvature at its nose R_B one obtains the following relations in "divergence" form:

$$(\partial/\partial s)[r^j t] + (\partial/\partial n)[(1 + n/R)r^j h] = 0 \quad (1)$$

$$(\partial/\partial s)[r^j Z] + (\partial/\partial n)[(1 + n/R)r^j H] - G = 0 \quad (2)$$

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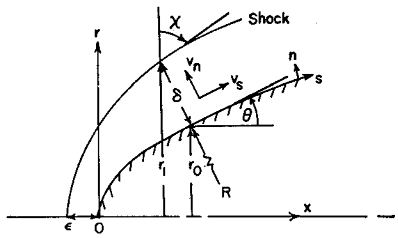

Fig. 1 Blunt body geometry.

where

$$t = \tau V_S,\; h = \tau V_n,\; Z = \rho V_S V_n$$

$$\tau = (1 - V^2)^{1/(\gamma-1)},\; H = \rho V_n{}^2 + kp,\; k = (\gamma - 1)/2\gamma$$

$$g = \rho V_S{}^2 + kp,\; G = (r^j/R)g + j(1 + n/R)kp\cos\theta$$

The velocities V are referred to the maximum adiabatic velocity, and the pressure and density, p and ρ, to their free-stream stagnation values. The parameter j is 0 or 1 for plane or axisymmetric flow, respectively.

The approximations are made as follows:

$$r^j t = r_0{}^j t_0 + (n/\delta)(r_1{}^j t_1 - r_0{}^j t_0)$$

$$r^j Z = (n/\delta)r_1{}^j Z_1,\; G = G_0 + (n/\delta)(G_1 - G_0)$$

$(\;)_0$ and $(\;)_1$ refer, respectively, to properties evaluated on the body and just behind the shock.

It should be noted here that with more than one strip, higher-order polynomials can be handled. Substituting the above in Eqs. (5) and (6) and taking

$$\int_0^\delta (\;)dn$$

(employing Leibnitz's rule), yields the method of integral relations. If instead one lets

$$\frac{\partial(\;)}{\partial n} = \frac{(\;)_1 - (\;)_0}{\delta},\; \frac{\partial(\;)}{\partial s} = \tfrac{1}{2}\left[\frac{d(\;)_1}{ds} + \frac{d(\;)_0}{ds}\right]$$

the method of lines is obtained. Through use of the oblique shock relations one can reduce the number of dependent variables to three, namely δ, χ, and V_{S_0}, where δ and χ are the local shock layer thickness and shock angle. The third equation is derived from the geometry of the shock. In either

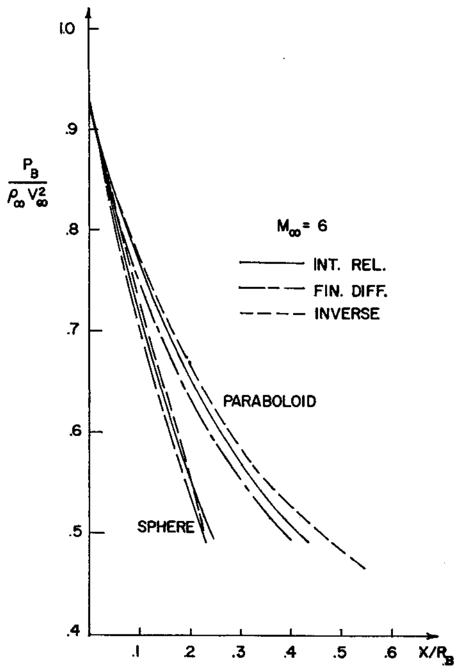

Fig. 2 Body pressure vs axial distance.

Table 1 Comparison of results $M_\infty = 6$, $\gamma = 1.4$

		Int. rel.	Meth. of lines	Inverse[6]
R_B/R_S,	Sphere	0.660	0.677	0.732
	Paraboloid	0.590	0.613	0.662
ϵ,	Sphere	0.148	0.145	0.148
	Paraboloid	0.164	0.158	0.163
χ^*/R_B,	Sphere	0.248	0.234	0.230
	Paraboloid	0.435	0.400	0.479

scheme, the resulting set of equations is conveniently expressed as

$$d\delta/ds = (1 + \delta/R)/\tan(\theta + \chi) \quad (3)$$

$$d\chi/ds = [(F_2 + Qf_1)(d\delta/ds) + F_1]/F_3 \quad (4)$$

$$dV_{S_0}/ds = [F_4 + (F_5 + Qf_2)(d\delta/ds) + F_6(d\chi/ds)]/F_7 = N/D \quad (5)$$

where the $F_i = F_i(s,\delta,\chi,V_{S_0})$ (see Ref. 5) and

$$f_1 = -\tfrac{1}{2}r_1{}^j Z_1, f_2 = \tfrac{1}{2}(r_0{}^j t_0 - r_1{}^j t_1)$$

and $Q = 1$ or 0 for the method of lines or integral relations, respectively.

Reference 5 provides an excellent review of the computational technique. The difference method exhibits the same qualitative behavior as does the integral relations, and consequently the computer program was able to apply the same numerical scheme to both, the only difference being the value of Q in the equations themselves. A Runge-Kutta technique was used to integrate numerically the equations from the axial streamline to the sonic point on the body. A step size of 0.02 was used for all cases; smaller steps were tried with negligible effect on the results. The usual computational problem arises because the equations are elliptic and nominally require that conditions be known along all four boundaries of the strip. Since the shock standoff distance ϵ and the location of the sonic point s^* are not known a priori, a value of ϵ is assumed at the start of each integration and an iterative approach is then followed. In the expression for dV_{S_0}/ds both N and D become identically zero at the sonic point. If the "correct" value of ϵ has been used, the ratio remains well-behaved and finite until $M = 1$ is achieved. However, the slightest deviation from this value (anything above $\sim 10^{-7}$) results in singular behavior of the gradient, and consequently a new ϵ is chosen and the integration begun again at the axis. Note that we treat here only smooth bodies with no discontinuity in slope. If such a "corner" does exist and as is often the case, it is known to coincide with the sonic point, discontinuous behavior in dV_S/ds is to be expected, and the convergence to the appropriate ϵ is greatly simplified.

Typical results are presented in Fig. 2 for both spherical and paraboloidal geometries at $M_\infty = 6$ and $\gamma = 1.4$ in the form of body pressures scaled to twice the freestream dynamic pressure; comparison is made with an inverse scheme by Van Dyke and Gordon.[6] Table 1 gives the corresponding values of shock standoff distance, ratio of body radius to shock radius of curvature at the nose, and location of sonic point on the body.

Other cases have been run for M_∞ ranging from 3–10^4. Generally speaking, the results of both single-strip approximations produce better agreement with the more precise inverse calculations as $M_\infty \to \infty$, which is not surprising in view of the narrowing shock layer. The integral approach is usually slightly more accurate as regards body pressures and detachment distances. As M_∞ enters the hypersonic range, for some geometries the method of lines becomes superior in predicting sonic point location. Since computing the shock radius of curvature involves approximating second-order derivatives, the single-strip methods provide rather poor agreement with Van Dyke's values of R_B/R_S, the method of lines again being closer at high Mach numbers.

In short then, both methods give reasonably good results, each perhaps having a slight advantage in different areas. However, as noted above the application of the method of lines with more than one strip is considerably simpler. In addition, the reluctance toward using it on grounds that many more strips are required to achieve a comparable degree of accuracy is apparently unfounded, at least as concerns the blunt body problem.

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Influence of Initial Flow Direction on the Turbulent Base Pressure in Supersonic Axisymmetric Flow

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Nomenclature

D = diameter
d,j = dividing and separating streamlines in Fig. 2
L,l = afterbody and total body lengths, respectively
M = Mach number
P = absolute pressure
R = reference streamline close to, but outside of, the mixing region

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